

Geological Shape Curvature

Principal normal curvatures, κ_1 and κ_2 , are defined at any point on a continuous surface such that:

$$\kappa_1 \geq \kappa_2, \quad -\infty \leq \kappa_1 \leq +\infty \text{ and } -\infty \leq \kappa_2 \leq +\infty \quad (1)$$

The Gaussian curvature, κ_G , and the mean normal curvature, κ_M , are defined in terms of the principal normal curvatures as:

$$\kappa_G = \kappa_1 \kappa_2, \quad \kappa_M = \frac{1}{2}(\kappa_1 + \kappa_2) \quad (2)$$

Eight geological shape curvature classes are defined in terms of the Gaussian and mean normal curvatures as follows:

$$\left. \begin{array}{l} \kappa_G < 0 \\ \kappa_M > 0 \end{array} \right\} \text{synformal saddle,} \quad \left. \begin{array}{l} \kappa_G = 0 \\ \kappa_M > 0 \end{array} \right\} \text{synform,} \quad \left. \begin{array}{l} \kappa_G > 0 \\ \kappa_M > 0 \end{array} \right\} \text{basin}$$

$$\left. \begin{array}{l} \kappa_G < 0 \\ \kappa_M = 0 \end{array} \right\} \text{perfect saddle,} \quad \left. \begin{array}{l} \kappa_G = 0 \\ \kappa_M = 0 \end{array} \right\} \text{plane} \quad (3)$$

$$\left. \begin{array}{l} \kappa_G < 0 \\ \kappa_M < 0 \end{array} \right\} \text{antiformal saddle,} \quad \left. \begin{array}{l} \kappa_G = 0 \\ \kappa_M < 0 \end{array} \right\} \text{antiform,} \quad \left. \begin{array}{l} \kappa_G > 0 \\ \kappa_M < 0 \end{array} \right\} \text{dome}$$

Each class is associated with a name that connotes an idealized shape and a color (Figure 1).

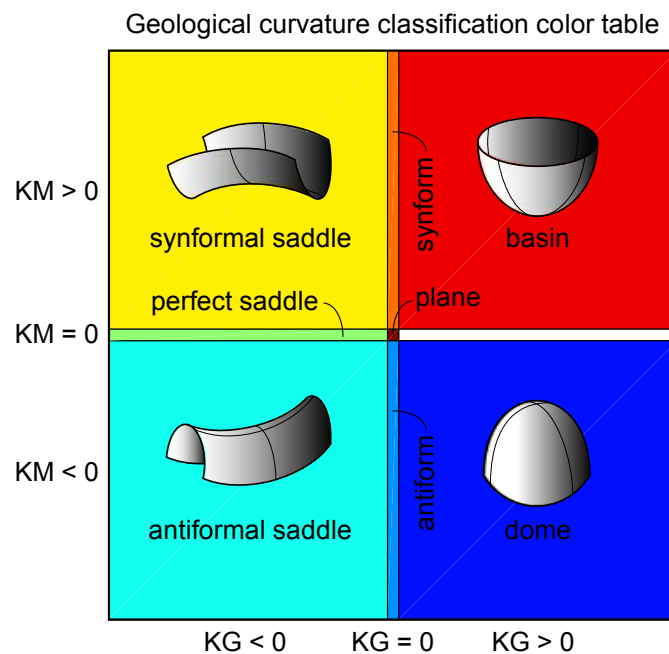


Figure 1. Color table for geological shape curvature. The narrow color bands labeled antiform and synform, and that labeled perfect saddle, represent exactly zero values of κ_M and κ_G respectively. The small colored square at the center labeled plane represents exactly zero values of both κ_M and κ_G . The narrow white band represents the mathematically inadmissible class.

The eight distinct colors are used to decorate a geological surface, thereby identifying the local shape of that surface (Figure 2). The data for Figure 2 are taken from Bellahsen et al., 2006 who digitized contour lines on the structure contour map of Forster et al., 1996 for the base of the Jurassic Sundance Formation at Sheep Mountain, Wyoming. There are likely to be substantial errors in elevation introduced during construction of the structure contour map and during gridding of the surface from the digitized contours. These errors have not been evaluated, so we take the surface as given and proceed with the curvature analysis of it as an illustration of the concepts and methods. We anticipate a future evaluation based upon comparison to a gridded surface generated by interpolation of Airborne Laser Swath Mapping (ALSM) data.

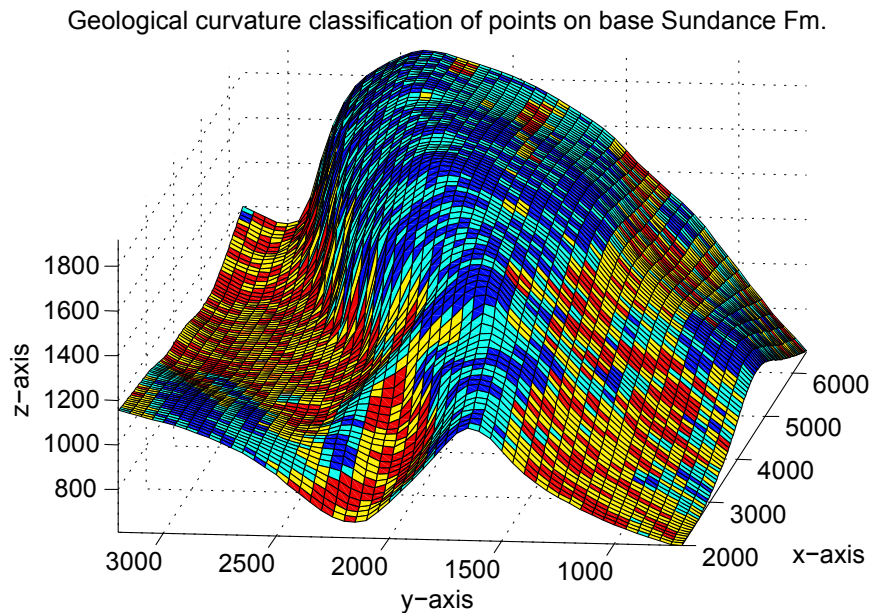


Figure 2. Geological shape curvature classification for base Sundance at Sheep Mountain, WY.

The MatLab code `surf_curv_pure.m` calculates and plots the surface in 3D space (x,y,z) , the principal normal curvature magnitudes, the principal normal curvature directions as vectors on tangent planes, and the geologic shape curvature classes. Also calculated are the coefficients of the first and second fundamental forms, and the shape operator. Curves that are concave upward (positive z) have positive curvature. The input for the code is three tab-delimited text files containing the gridded x , y , and z coordinates, respectively, of points on the surface. The grid of x - and y -coordinates are organized in these files as illustrated in (4). Each elevation (z -value) in the third file is associated with the corresponding pair of (x, y) coordinates in the first two files.

$$\begin{bmatrix} \min(x) & \min(x)+dx & \cdots & \max(x) \\ \min(x) & \min(x)+dx & \cdots & \max(x) \\ \vdots & & & \vdots \\ \min(x) & & & \max(x) \end{bmatrix} \begin{bmatrix} \min(y) & \min(y) & \cdots & \min(y) \\ \min(y)+dy & \min(y)+dy & \cdots & \min(y)+dy \\ \vdots & & & \vdots \\ \max(y) & & & \max(y) \end{bmatrix} \quad (4)$$

The definitions given in (3) lead to certain mathematical and data analysis issues such that the values of Gaussian and mean normal curvatures, calculated from coordinate data on a geological surface, result in a preponderance of points falling into the classes in the four corners of the color table (Figure 1). This phenomenon is apparent for the Sundance surface from Sheep Mountain anticline when it is rotated so the view is directly downward (Figure 3). The numbers of points in each class are: 1375 antiformal saddle; 0 antiform, 1125 dome; 0 perfect saddle; 2 plane; 1283 synformal saddle; 20 synform; and 836 basin.

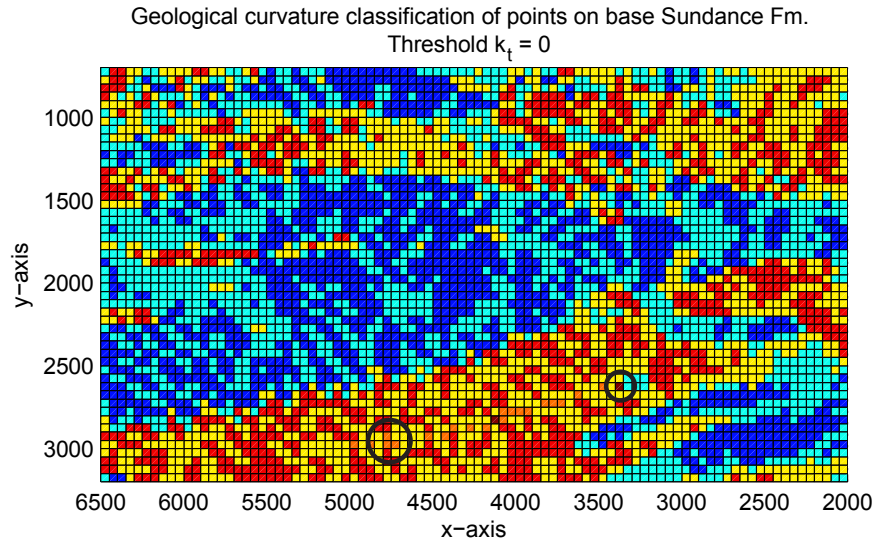


Figure 3. Geological shape curvature classification with zero threshold. The smaller black circle locates 1 of 2 planar points; the larger black circle locates 3 of 20 synformal points.

Consider the Gaussian and mean normal curvatures for antiformal and synformal points:

$$\kappa_G = 0 \text{ and } \kappa_M \neq 0 \text{ requires } \kappa_1 = 0 \text{ or } \kappa_2 = 0$$

Apparently it is unlikely that a calculated value of principal normal curvature is exactly zero for a natural surface. Furthermore, given the inevitable errors in measurements of coordinate data, it would be difficult to defend a value as being exactly zero. Consider the perfect saddle:

$$\kappa_G < 0 \text{ and } \kappa_M = 0 \text{ requires } \kappa_1 = -\kappa_2$$

It is unlikely that calculated values of principal normal curvatures would be exactly equal in magnitude and opposite in sign. Again, errors in measurement preclude a defense of this special case. Consider the plane:

$$\kappa_G = 0 \text{ and } \kappa_M = 0 \text{ requires } \kappa_1 = 0 = \kappa_2$$

The arguments given above apply to this case. Finally, consider the unlabeled class shown as the narrow white band on Figure 1:

$$\kappa_G > 0 \text{ and } \kappa_M = 0 \text{ requires } \kappa_1 = -\kappa_2 \neq 0$$

This is mathematically inadmissible because the product of two numbers, equal in magnitude and opposite in sign, always is less than zero.

It is possible to address the inevitable errors introduced in measurements of coordinate data on geological surfaces, and to reconcile the problem that natural surfaces have a preponderance of points classified into the corner positions of the color table, by introducing the concept of a normal curvature threshold. If the magnitude of a principal normal curvature is less than or equal to a positive constant called the threshold, k_t , then that curvature is set to zero:

$$\left. \begin{array}{l} \text{if } |\kappa_1| \leq k_t \text{ then } \kappa_1 = 0 \\ \text{if } |\kappa_2| \leq k_t \text{ then } \kappa_2 = 0 \end{array} \right\} \text{ for } k_t \geq 0 \quad (5)$$

This redefinition restricts the ranges of the principal normal curvatures such that:

$$\kappa_1 \begin{cases} > k_t \\ = 0 \\ < -k_t \end{cases}, \quad \kappa_2 \begin{cases} > k_t \\ = 0 \\ < -k_t \end{cases} \quad (6)$$

The consequences for the Gaussian curvature are found by writing down a table of all possible values of $\kappa_1 \kappa_2$ using the ranges in (6):

	$\kappa_2 > k_t$	$= 0$	$< -k_t$
$\kappa_1 > k_t$	$> k_t^2$	$= 0$	$< -k_t^2$
$= 0$		$= 0$	$= 0$
$< -k_t$			$> k_t^2$

The cases in the lower left of this table are ruled out because $\kappa_1 \geq \kappa_2$. The restrictions imposed by (6) on the Gaussian curvature are related to the square of the threshold. The consequences for the mean normal curvature are found by writing down all possible values of $(\kappa_1 + \kappa_2)/2$ using the ranges in (6):

	$\kappa_2 > k_t$	$= 0$	$< -k_t$
$\kappa_1 > k_t$	$> k_t$	$> \frac{1}{2}k_t$	$\geq -\infty, \leq +\infty$
$= 0$		$= 0$	$< -\frac{1}{2}k_t$
$< -k_t$			$< -k_t$

Again, the cases in the lower left of this table are ruled out because $\kappa_1 \geq \kappa_2$. Also, the unrestricted range for the mean normal curvature (upper right case) is found if the principal curvatures have opposite signs and unequal magnitudes.

In summary, using a threshold to restrict the ranges of the principal normal curvatures as in (6), also restricts the range of the Gaussian curvature, but does not restrict the range of the mean normal curvature:

$$\kappa_G \begin{cases} > k_t^2 \\ = 0 \\ < -k_t^2 \end{cases}, \quad -\infty \leq \kappa_M \leq +\infty \quad (7)$$

For the synformal saddle with a threshold imposed on the principal normal curvatures we have $\kappa_1 > k_t$ and $\kappa_2 < -k_t$. Because the range of the mean normal curvature is not restricted by these conditions, we must impose the threshold on it, such that $\kappa_M > k_t$. For the cylindrical synform with a threshold we have $\kappa_1 > k_t$ and $\kappa_2 = 0$. The consequence for the mean normal curvature is $\kappa_M > \frac{1}{2}k_t$. For the basin with a threshold we have $\kappa_1 > k_t$ and $\kappa_2 > k_t$ so the range of mean normal curvatures is restricted such that $\kappa_M > k_t$. Using similar arguments the ranges for the antiformal saddle, antiform, and dome are redefined. Together these new ranges serve to redefine the ranges for the perfect saddle and plane. The eight geological shape curvature classes are:

$$\begin{aligned} & \left. \begin{array}{l} \kappa_G < -k_t^2 \\ \kappa_M > k_t \end{array} \right\} \text{synformal saddle,} & \left. \begin{array}{l} \kappa_G = 0 \\ \kappa_M > \frac{1}{2}k_t \end{array} \right\} \text{synform,} & \left. \begin{array}{l} \kappa_G > k_t^2 \\ \kappa_M > k_t \end{array} \right\} \text{basin} \\ & \left. \begin{array}{l} \kappa_G < -k_t^2 \\ |\kappa_M| \leq k_t \end{array} \right\} \text{perfect saddle,} & \left. \begin{array}{l} \kappa_G = 0 \\ |\kappa_M| \leq \frac{1}{2}k_t \end{array} \right\} \text{plane} & \\ & \left. \begin{array}{l} \kappa_G < -k_t^2 \\ \kappa_M < -k_t \end{array} \right\} \text{antiformal saddle,} & \left. \begin{array}{l} \kappa_G = 0 \\ \kappa_M < -\frac{1}{2}k_t \end{array} \right\} \text{antiform,} & \left. \begin{array}{l} \kappa_G > k_t^2 \\ \kappa_M < -k_t \end{array} \right\} \text{dome} \end{aligned} \quad (8)$$

The color table corresponding to the ranges in (8) is provided as Figure 4.

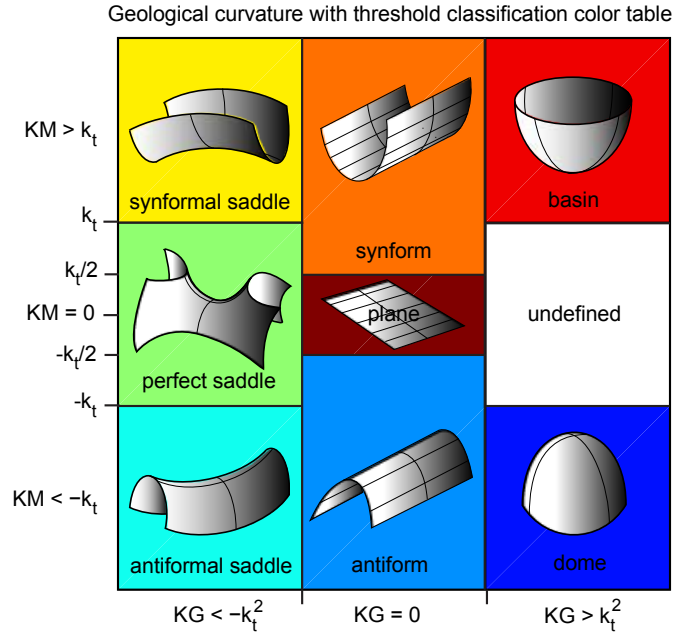


Figure 4. Color table for geological shape curvature with a threshold k_t operating on both principal normal curvatures according to (5).

The ranges of principal normal curvatures for the base of the Sundance formation with no threshold (Figure 2) are:

$$-0.0021 \leq \kappa_1 \leq 0.0105 \text{ and } -0.0112 \leq \kappa_2 \leq 0.0019 \quad (9)$$

A meaningful threshold for this surface must be less than the greatest magnitude of either principal curvature, so $k_t < 0.0112$. How is the value chosen? The criterion suggested here relates the threshold to the standard error σ associated with the elevation data.

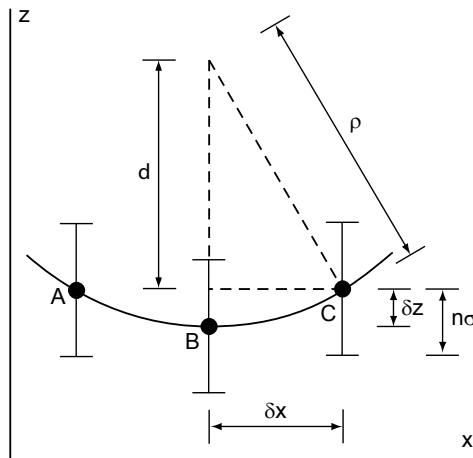


Figure 5. Geometric relations used to define the curvature threshold.

Referring to Figure 5 consider three successive data points (A, B, and C) in the x -coordinate direction with constant spacing δx . A circular arc is drawn through these points with radius ρ . The distance d is measured from the center of the circular arc to the level line connecting points A and C, and the elevation change from that line to the point B is δz . From Pythagorus we have:

$$\rho^2 = (\delta x)^2 + d^2$$

Substituting for $d = \rho - \delta z$ and rearranging to solve for ρ , the radius of curvature is:

$$\rho = \frac{(\delta x)^2 + (\delta z)^2}{2\delta z}$$

To distinguish the arc ABC from the straight line AC we assert that the elevation change δz must exceed some multiple of the standard error of the elevation measurements, $n\sigma$. Noting that the curvature is the reciprocal of the radius of curvature and substituting the multiple of the standard error for the elevation change, we define the threshold:

$$k_t = 2n\sigma / [(\delta x)^2 + (n\sigma)^2] \quad (10)$$

Calculated values of normal curvature that are less than or equal to the threshold signal that the elevation data are not precise enough to defend a non-zero curvature. This methodology may be extended to evaluate normal curvatures in any direction on a surface, including the principal directions.

For example, suppose that the standard error for the gridded elevation data for the Sundance formation at Sheep Mountain is $\sigma = 1$ m. Using (10) with $n = 1$ we find the corresponding threshold $k_t = 0.0008$, which corresponds to a radius of curvature of 1,250m. This threshold is well within the range of magnitudes of the principal curvatures (9). As in Figure 2 the geological shape curvature is computed, but now it is constrained by the threshold as implemented in the MatLab script `surf_curv_thres.m` (Figure 6). On the Sundance surface the numbers of points in each class are: 44 antiformal saddle; 1,191 antiform, 68 dome; 94 perfect saddle; 2,148 plane; 36 synformal saddle; 988 synform; and 72 basin. The shape curvature at 46% of the 4,641 grid points on the surface is planar. That is, the Sundance formation locally is indistinguishable from a plane at these points, given the precision of the elevation data. Cylindrical antiforms and synforms comprise 26% and 21% of the points respectively, and are concentrated in the two hinges of the folded Sundance surface.

The shape curvatures are best illustrated for the Sundance surface from Sheep Mountain when it is rotated so the view is directly downward (Figure 7). Comparison to the shape curvatures for the surface with zero threshold (Figure 3) emphasizes the fact that for most of the more gently dipping back limb the shape cannot be distinguished from planar given a 1m standard error in elevations. A similar conclusion may be reached for the much less extensive

forelimb. To address questions related to the curvature of exposed stratigraphic surfaces in the back- and forelimbs more precise elevation data would be essential. The majority of points in the crest of the fold are antiformal in shape, with some domal and antiformal saddle points. The majority of points in the trough of the fold are synformal in shape, but little of the stratigraphy is exposed in this region so the elevation errors are probably quite significant.

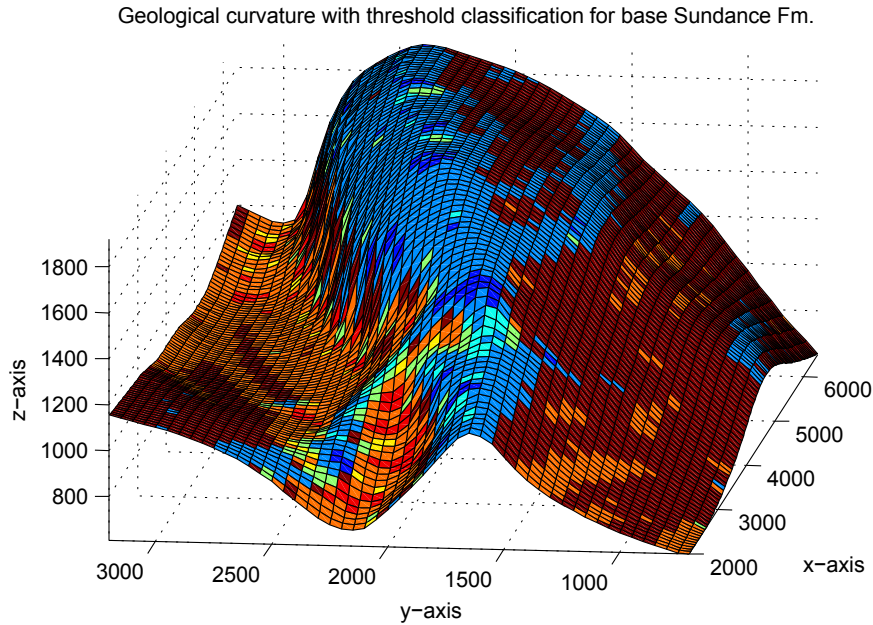


Figure 6. Geological shape curvature classification with a threshold, $k_t = 0.0008$, for the base Sundance at Sheep Mountain. See Figure 4 for color table.

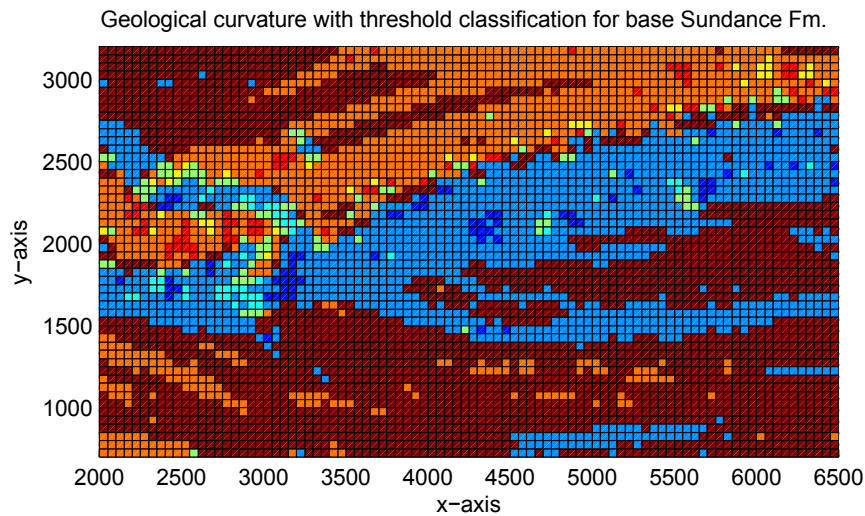


Figure 7. Geological shape curvature classification for Sheep Mountain anticline with a non-zero threshold, $k_t = 0.0008$, associated with a standard error $\sigma = 1m$.